Strict inequality in the box-counting dimension product formulas

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ABSTRACT

It is known that the upper box-counting dimension of a Cartesian product satisfies the inequality $\dim_B (F \times G) \leq \dim_B (F) + \dim_B (G)$ whilst the lower box-counting dimension satisfies the inequality $\dim_{LB} (F \times G) \geq \dim_{LB} (F) + \dim_{LB} (G)$. We construct Cantor-like sets to demonstrate that both of these inequalities can be strict.

1. Preliminaries

In a metric space X the Hausdorff dimension of a compact set $F \subset X$ is defined as the supremum of the $d \geq 0$ such that the d-dimensional Hausdorff measure $\mathcal{H}^d (F)$ of F is infinite. The Hausdorff dimension takes values in the non-negative reals and extends the elementary integer-valued topological dimension in the sense that for a large class of 'reasonable' sets these two values coincide. Sets with non-coinciding Hausdorff and topological dimensions are called 'fractal', a term coined by Mandelbrot in his original study of such sets [**8**]. Hausdorff introduced this generalised dimension in [**6**] and its subsequent extensive use in geometric measure theory is developed by Federer [**5**] and Falconer [**3**]. The fact that the Hausdorff dimension satisfies $\dim_H (F \times G) \geq \dim_H (F) + \dim_H (G)$ for the Cartesian product of sets was proved in full generality in [**7**] (and later summarised in [**4**] §7.1 'Product Formulae') after some partial results: The inequality was proved in [**1**] with the restriction that $0 < \mathcal{H}^s (F), \mathcal{H}^t (G) < \infty$ for some s, t and was extended to a larger class of sets in [**2**]. The paper [**1**] also provides an example for which there is a strict inequality in the product formula and again this is summarised in [**4**] §7.1.

In this paper we prove similar product inequalities for the upper and lower box-counting dimensions which are less familiar generalisations of dimension (treated briefly in [**4**]; see [**10**] for a more detailed exposition) and have applications to dynamical systems (see, for example, [**9**]). Our main result is an example analogous to that in [**1**] which demonstrates that the box-counting product inequalities can be strict. In a metric space X the upper and lower box-counting dimensions of a compact set $F \subset X$ are defined by

$$\dim_B (F) = \limsup_{\delta \searrow 0} \frac{\log (N (F, \delta))}{- \log \delta} \tag{1.1}$$

and

$$\dim_{LB} (F) = \liminf_{\delta \searrow 0} \frac{\log (N (F, \delta))}{- \log \delta} \tag{1.2}$$

respectively, where $N (F, \delta)$ is the smallest number of sets with diameter at most δ which form a cover (called a δ-cover) of F. Essentially, if $N (F, \delta)$ scales like $\delta^{-\varepsilon}$ as $\delta \to 0$ then these quantities capture ε which gives an indication of how many more sets are required to cover

F as the length-scales decrease and so encodes how 'spread out' the set F is at small length-scales. These limits are unchanged if we replace $N(F, \delta)$ with one of many similar quantities (discussed by Falconer in [**4**] §3.1 'Equivalent Definitions'). Of these quantities we will also make use of the largest number of disjoint closed balls of diameter δ with centres in F, which we denote $M(F, \delta)$.

We first recall the standard (see, for example, [**4**] or [**10**]) proof of the box-counting product inequalities when F and G are compact sets in metric spaces X and Y respectively, although the inequality (1.4) is less familiar (Robinson provides a proof in [**10**]). We endow the product space $X \times Y$ with the usual Euclidean metric $d_{X \times Y} = \sqrt{d_X^2 + d_Y^2}$, but the proof can be adapted for a variety of product metrics (see [**10**]).

THEOREM 1.1. *For compact sets $F \subset X$ and $G \subset Y$ the box-counting dimensions of the product set $F \times G$ satisfy the inequalities*

$$\dim_B (F \times G) \leq \dim_B (F) + \dim_B (G) \tag{1.3}$$
$$\dim_{LB} (F \times G) \geq \dim_{LB} (F) + \dim_{LB} (G) \tag{1.4}$$

Proof. Suppose $\{U_i\}_{i=1}^{n_1}$ and $\{V_j\}_{j=1}^{n_2}$ are δ-covers of F and of G respectively then the set of products $\{U_i \times V_j | i = 1 \ldots n_1, j = 1 \ldots n_2\}$ is a cover of $F \times G$ with a total of $n_1 n_2$ elements and the diameter of each $U_i \times V_j$ is no greater than $\sqrt{2}\delta$. As there exist δ-covers of F and G consisting of $N(F, \delta)$ and $N(G, \delta)$ elements respectively this construction gives a $\sqrt{2}\delta$-cover of $F \times G$ consisting of $N(F, \delta) N(G, \delta)$ elements, hence

$$N\left(F \times G, \sqrt{2}\delta\right) \leq N(F, \delta) N(G, \delta). \tag{1.5}$$

Next, if both $\{x_i\}_{i=1}^{n_1} \subset F$ and $\{y_j\}_{j=1}^{n_2} \subset G$ are sets of centres of disjoint balls with diameter δ then the balls with radius δ centred on the $n_1 n_2$ points $\{(x_i, y_j) | i = 1 \ldots n_1, j = 1 \ldots n_2\} \subset F \times G$, are also disjoint. As there exist sets of disjoint balls of diameter δ with centres in F and G consisting of $M(F, \delta)$ and $M(G, \delta)$ elements respectively the above construction gives $M(F, \delta) M(G, \delta)$ disjoint balls of diameter δ with centres in $F \times G$, hence

$$M(F \times G, \delta) \geq M(F, \delta) M(G, \delta). \tag{1.6}$$

From (1.1) and (1.5) we see that

$$\begin{aligned}
\dim_B (F \times G) &= \limsup_{\delta \searrow 0} \frac{\log\left(N\left(F \times G, \sqrt{2}\delta\right)\right)}{-\log\left(\sqrt{2}\delta\right)} \\
&\leq \limsup_{\delta \searrow 0} \left[\frac{\log(N(F, \delta))}{-\log\left(\sqrt{2}\delta\right)} + \frac{\log(N(G, \delta))}{-\log\left(\sqrt{2}\delta\right)}\right] \\
&\leq \limsup_{\delta \searrow 0} \frac{\log(N(F, \delta))}{-\log \delta - \log \sqrt{2}} + \limsup_{\delta \searrow 0} \frac{\log(N(G, \delta))}{-\log \delta - \log \sqrt{2}} \\
&= \dim_B (F) + \dim_B (G).
\end{aligned} \tag{1.7}$$

From (1.2) and (1.6) we have

$$
\begin{aligned}
\dim_{LB}(F \times G) &= \liminf_{\delta \searrow 0} \frac{\log\left(M\left(F \times G, \delta\right)\right)}{-\log \delta} \\
&\geq \liminf_{\delta \searrow 0} \left[\frac{\log\left(M\left(F, \delta\right)\right)}{-\log \delta} + \frac{\log\left(M\left(G, \delta\right)\right)}{-\log \delta} \right] \\
&\geq \liminf_{\delta \searrow 0} \frac{\log\left(M\left(F, \delta\right)\right)}{-\log \delta} + \liminf_{\delta \searrow 0} \frac{\log\left(M\left(G, \delta\right)\right)}{-\log \delta} \\
&= \dim_{LB}(F) + \dim_{LB}(G).
\end{aligned}
\tag{1.8}
$$

$\square$

It is known that there are sets with unequal upper and lower box-counting dimension (see exercise 3.8 of [**4**] or [**10**] §3.1), however if these values coincide for a set F we define their common value as the box-counting dimension of F. If sets F and G have well-defined box-counting dimension then the box-counting dimension of their product is also well behaved.

COROLLARY 1.2.   If $\dim_B(F) = \dim_{LB}(F)$ and $\dim_B(G) = \dim_{LB}(G)$ then

$$
\dim_B(F \times G) = \dim_{LB}(F \times G) = \dim_B(F) + \dim_B(G).
$$

Proof.   From the inequalities (1.3) and (1.4) we have

$$
\dim_B(F \times G) \leq \dim_B(F) + \dim_B(G) = \dim_{LB}(F) + \dim_{LB}(G) \leq \dim_{LB}(F \times G)
$$

but from the definition of the box-counting dimension $\dim_{LB}(F \times G) \leq \dim_B(F \times G)$ so we must have equality throughout the above. $\square$

In the following construction both sets F and G have non-coinciding upper and lower box-counting dimensions so that as $\delta \to 0$ the box-counting functions $\frac{\log(N(F,\delta))}{-\log \delta}$ and $\frac{\log(N(G,\delta))}{-\log \delta}$ oscillate between two values. Further, by ensuring that these functions oscillate with different phases (see figure 1) we can produce strict inequalities after (1.7) and (1.8) and so yield strict inequality in both product formulas, that is

$$
\dim_{LB}(F \times G) < \dim_{LB}(F) + \dim_{LB}(G) < \dim_B(F) + \dim_B(G) < \dim_B(F \times G).
$$

To this end we construct variations of the Cantor middle-third set from the initial interval $[0, 1]$ except at each stage we use one of the three generators $\text{gen}_3, \text{gen}_5$ or gen_7 which remove the middle $\frac{1}{3}, \frac{3}{5}$ or $\frac{5}{7}$ of each interval respectively. Note that if we exclusively use the gen_3 generator we produce the usual Cantor middle-third set, which has lower and upper box-counting dimension $\frac{\log(2)}{\log(3)}$ and if we exclusively use the gen_7 generator we produce a similar Cantor set with lower and upper box-counting dimension $\frac{\log(2)}{\log(7)}$. By switching generators at certain stages of our construction we can cause $\frac{\log(N(F,\delta))}{-\log \delta}$ to oscillate between these values, providing that we apply the generators a sufficiently large number of times.
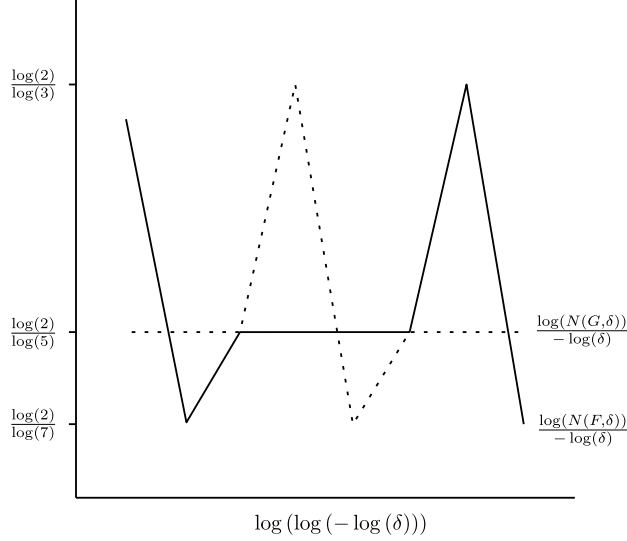
FIGURE 1. *The box-counting functions for the sets F and G constructed in the next section (explicitly computed here for small δ) oscillate between $\frac{\log(2)}{\log(3)}$ and $\frac{\log(2)}{\log(7)}$. The x-axis is scaled so that the slow oscillation can be graphed, and this oscillation continues as $\log\left(\log\left(-\log\left(\delta\right)\right)\right) \to \infty$, that is as $\delta \to 0$. The differing phases guarantee that the sum of these functions doesn't approach either $\frac{\log(2)}{\log(3)} + \frac{\log(2)}{\log(3)}$ or $\frac{\log(2)}{\log(7)} + \frac{\log(2)}{\log(7)}$.*

To simplify notation, we choose a sequence of integers $K_j = 10^{2^j}$ which increases sufficiently quickly that

$$\sum_{i=0}^{j} K_i < K_{j+1} \tag{1.9}$$

$$\sum_{i=0}^{j} K_i - K_{i-1} > K_{j-1} \tag{1.10}$$

$$\frac{\log\left(7\right)}{\log\left(3\right)} K_j < K_{j+1} \tag{1.11}$$

and

$$\frac{\sum_{i=0}^{j-1} K_i}{K_j} \to 0 \quad \text{as } j \to \infty \tag{1.12}$$

2. Constructing sets F and G

We construct two sets F and G using the following iterative procedure: For F apply the following generator at the jth stage

$$\begin{cases} \text{gen}_3 & K_{6n} < j \leq K_{6n+1} \quad \text{for some} \quad n \in \mathbb{N} \\ \text{gen}_7 & K_{6n+1} < j \leq K_{6n+2} \quad \text{for some} \quad n \in \mathbb{N} \\ \text{gen}_5 & \text{otherwise,} \end{cases}$$

and for G apply the following generator at the jth stage

$$\begin{cases} \text{gen}_3 & K_{6m+3} < j \le K_{6m+4} \quad \text{for some} \quad m \in \mathbb{N} \\ \text{gen}_7 & K_{6m+4} < j \le K_{6m+5} \quad \text{for some} \quad m \in \mathbb{N} \\ \text{gen}_5 & \text{otherwise.} \end{cases}$$

Let $f_3(j)$ be the number of times gen_3 has been applied and $f_7(j)$ the number of times gen_7 has been applied in the construction F by stage j. With this notation gen_5 has been applied $j - f_3(j) - f_7(j)$ times by stage j. Similarly define $g_3(j)$ and $g_7(j)$ for the construction of G. Clearly these functions are non-decreasing and we refrain from writing them explicitly except to note that

$$f_3(K_{6n+1}) = \sum_{i=0}^{n} K_{6i+1} - K_{6i} \qquad f_7(K_{6n+2}) = \sum_{i=0}^{n} K_{6i+2} - K_{6i+1}$$
$$g_3(K_{6m+4}) = \sum_{i=0}^{m} K_{6i+4} - K_{6i+3} \quad g_7(K_{6m+5}) = \sum_{i=0}^{m} K_{6i+5} - K_{6i+4} \tag{2.1}$$

and

$$\begin{aligned} f_3(j) &= f_3(K_{6n+1}) & \text{for} && K_{6n+1} < j \le K_{6n+6} \\ f_7(j) &= f_7(K_{6n+2}) & \text{for} && K_{6n+2} < j \le K_{6n+7} \\ g_3(j) &= g_3(K_{6m+3}) & \text{for} && K_{6m+3} < j \le K_{6m+8} \\ g_7(j) &= g_7(K_{6m+3}) & \text{for} && K_{6m+4} < j \le K_{6m+9} \end{aligned} \tag{2.2}$$

Denote the sets at the jth stage of the construction of F and G by

F_j, which consists of 2^j intervals of length $3^{-f_3(j)}7^{-f_7(j)}5^{-j+f_3(j)+f_7(j)}$ and

G_j, which consists of 2^j intervals of length $3^{-g_3(j)}7^{-g_7(j)}5^{-j+g_3(j)+g_7(j)}$

so that F and G are defined by $F = \bigcap F_j$ and $G = \bigcap G_j$. Note that for every j the endpoints of the intervals in F_j and G_j are in F and G respectively as the generators only remove the middle of each interval.

PROPOSITION 2.1.   *For δ such that*

$$3^{-f_3(j)}7^{-f_7(j)}5^{-j+f_3(j)+f_7(j)} \le \delta < 3^{-f_3(j-1)}7^{-f_7(j-1)}5^{-(j-1)+f_3(j-1)+f_7(j-1)} \tag{2.3}$$

we have $N(F, \delta) = M(F, \delta) = 2^j$.

We refer to those δ in the range (2.3) as length-scales corresponding to stage j in the construction of F. Clearly every $1 > \delta > 0$ is a length scale corresponding to exactly one stage j_δ and $j_\delta \to \infty$ as $\delta \to 0$. We refer to length-scales corresponding to the construction of G in an analogous fashion.

Proof. For δ in the range (2.3) the obvious cover consisting of all intervals in F_j gives $N(F, \delta) \le 2^j$. The opposite inequality comes from the fact that a set with diameter δ in this range intersects at most one $(j-1)^{\text{th}}$ stage interval I but cannot cover both j^{th} stage subintervals of I (see figure 2) so at least $2 \times 2^{j-1} = 2^j$ elements are needed to form a cover of F .

Next, δ in the range (2.3) is less than the length of the intervals in F_{j-1} so balls of diameter δ centred on the end points of the intervals of F_{j-1} are disjoint and have centres in F (see figure 3). This gives two disjoint balls for each interval, hence $M(F, \delta) \ge 2 \times 2^{j-1} = 2^j$. For the opposite inequality suppose for a contradiction that $M(F, \delta) > 2^j$, then at least one of the 2^{j-1} intervals in F_{j-1} contains the centres of least three disjoint balls with centres in F. Let
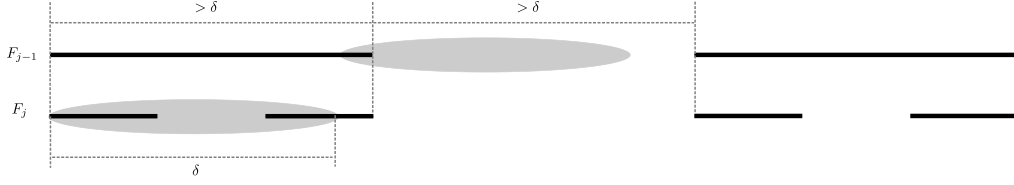
FIGURE 2. *A section of the sets F_{j-1} and F_j (shown in black) to illustrate that each set (shown as grey ellipses) with diameter $\delta < 3^{-f_3(j-1)} 7^{-f_7(j-1)} 5^{-(j-1)+f_3(j-1)+f_7(j-1)}$ (i.e. less than the length of the intervals of F_{j-1}) can neither intersect two intervals of F_{j-1} nor cover two intervals of F_j.*
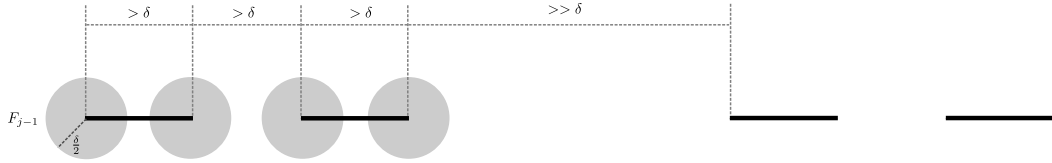

FIGURE 3. *A section of the set F_{j-1} (shown in black) to illustrate that balls (shown in grey) with diameter $\delta < 3^{-f_3(j-1)} 7^{-f_7(j-1)} 5^{-(j-1)+f_3(j-1)+f_7(j-1)}$ (i.e. less than the length of the intervals of F_{j-1}) with centres the endpoints of the intervals F_{j-1} are disjoint, giving two disjoint balls for each interval of F_{j-1}.*

I be such an interval. At the next stage of the construction I is split into two sub-intervals, one of which contains the centres of at least two of these three disjoint balls. However, this j^{th} stage subinterval has length no greater than δ so two closed balls of diameter δ centred in this interval cannot be disjoint (see figure 4), which is a contradiction. $\qquad\square$

Adapting the argument we can prove a similar proposition for the set G.

PROPOSITION 2.2.   *For δ such that*

$$3^{-g_3(j)} 7^{-g_7(j)} 5^{-j+g_3(j)+g_7(j)} \leq \delta < 3^{-g_3(j-1)} 7^{-g_7(j-1)} 5^{-(j-1)+g_3(j-1)+g_7(j-1)} \qquad (2.4)$$

we have $N(G, \delta) = M(G, \delta) = 2^j$.

By taking logarithms we obtain the following more useful form of propositions 2.1 and 2.2

$$f_3(j-1)[\log(3) - \log(5)] + f_7(j-1)[\log(7) - \log(5)] + (j-1)\log(5)$$
$$< -\log\delta \leq f_3(j)[\log(3) - \log(5)] + f_7(j)[\log(7) - \log(5)] + j\log(5)$$
$$\Rightarrow \log(N(F, \delta)) = \log(M(F, \delta)) = j\log(2) \quad (2.5)$$

and

$$g_3(j-1)[\log(3) - \log(5)] + g_7(j-1)[\log(7) - \log(5)] + (j-1)\log(5)$$
$$< -\log\delta \leq g_3(j)[\log(3) - \log(5)] + g_7(j)[\log(7) - \log(5)] + j\log(5)$$
$$\Rightarrow \log(N(G, \delta)) = \log(M(G, \delta)) = j\log(2) \quad (2.6)$$
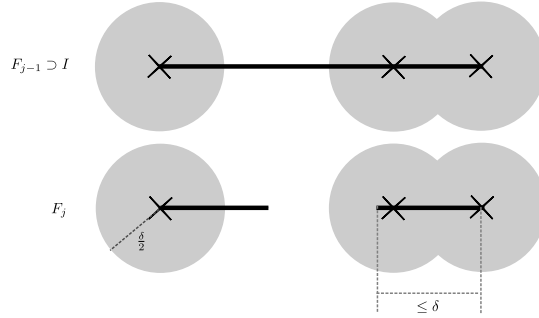
FIGURE 4. *A sub-interval $I \subset F_{j-1}$ (shown in black) which contains the centres of three balls (shown in grey) with diameter $\delta \geq 3^{-f_3(j)}7^{-f_7(j)}5^{-j+f_3(j)+f_7(j)}$ (i.e. greater than the the length of the intervals of F_j) with centres in F. As the centres are also contained in F_j (also shown in black) at least one interval of F_j contains the centres of two of these balls. Consequently, the distance between the centres is at most the length of an interval of F_j which is at most δ which is the sum of the radii of the closed balls so the two balls are not disjoint.*

The essential feature of our construction is that the sets F and G at some length-scales look like the Cantor middle-third set, while at other length-scales look like the Cantor middle-$\frac{5}{7}$th set. Further, this 'local' behaviour is maintained over sufficient length-scales that the box-counting limits of F and G at these length-scales approach the box-counting dimensions of the relevant Cantor set, which we will establish in the following section. We conclude this section by proving that at any length-scale the sets F and G do not both look like the middle-third set nor do they both look like the middle-$\frac{5}{7}$th set.

LEMMA 2.3. *No δ is both a length-scale corresponding to some stage in $(K_{6n}, K_{6n+2}]$ in the construction of F for some $n \in \mathbb{N}$ and a length-scale corresponding to some stage in $(K_{6m+3}, K_{6m+5}]$ in the construction of G for some $m \in \mathbb{N}$.*

Proof.   Assume for a contradiction that δ is such a length-scale, that is

$$3^{-f_3(K_{6n+2})}7^{-f_7(K_{6n+2})}5^{-K_{6n+2}+f_3(K_{6n+2})+f_7(K_{6n+2})}$$
$$\leq \delta < 3^{-f_3(K_{6n})}7^{-f_7(K_{6n})}5^{-K_{6n}+f_3(K_{6n})+f_7(K_{6n})} \quad (2.7)$$

and

$$3^{-g_3(K_{6m+5})}7^{-g_7(K_{6m+5})}5^{-K_{6m+5}+g_3(K_{6m+5})+g_7(K_{6m+5})}$$
$$\leq \delta < 3^{-g_3(K_{6m+3})}7^{-g_7(K_{6m+3})}5^{-K_{6m+3}+g_3(K_{6m+3})+g_7(K_{6m+3})} \quad (2.8)$$

We first demonstrate that this could only hold if $n = m$. From (2.7) and (2.8) we have $7^{-K_{6n+2}} \leq \delta < 3^{-K_{6n}}$ and $7^{-K_{6m+5}} \leq \delta < 3^{-K_{6m}}$ respectively, which in turn yield $7^{-K_{6n+2}} < 3^{-K_{6m}}$ and $7^{-K_{6m+5}} < 3^{-K_{6n}}$. Taking logarithms we get

$$K_{6m} < K_{6n+2}\frac{\log(7)}{\log(3)} < K_{6n+3} \quad \text{and} \quad K_{6n} < K_{6m+5}\frac{\log(7)}{\log(3)} < K_{6m+6}$$

where the final inequalities follow from the growth property (1.11). We conclude that $6m < 6n + 3$ and $6n < 6m + 6$, which implies that $m = n$. With this restriction, if δ is such a length-scale then the lower bound from (2.7) and the upper bound from (2.8) imply

$$3^{-f_3(K_{6n+2})}7^{-f_7(K_{6n+2})}5^{-K_{6n+2}+f_3(K_{6n+2})+f_7(K_{6n+2})}$$
$$< 3^{-g_3(K_{6n+3})}7^{-g_7(K_{6n+3})}5^{-K_{6n+3}+g_3(K_{6n+3})+g_7(K_{6n+3})}$$

For clarity we suppress the argument of the functions and take logarithms which yields

$$[\log(5) - \log(3)](f_3 - g_3) + [\log(5) - \log(7)](f_7 - g_7) + \log(5)(K_{6n+3} - K_{6n+2}) < 0. \quad (2.9)$$

The bounds in (1.9) and (1.10) give

$$g_3(K_{6n+3}) - f_3(K_{6n+2}) = \sum_{i=0}^{n-1} K_{6i+4} - K_{6i+3} - \sum_{i=0}^{n} K_{6i+1} - K_{6i}$$
$$< K_{6n-1} - K_{6n} < 0 \quad (2.10)$$

and

$$f_7(K_{6n+2}) - g_7(K_{6n+3}) = \sum_{i=0}^{n} K_{6i+2} - K_{6i+1} - \sum_{i=0}^{n-1} K_{6i+5} - K_{6i+4}$$
$$< K_{6n+3} - K_{6n-2}. \quad (2.11)$$

Consequently we drop the positive first term of (2.9) which implies

$$[\log(5) - \log(7)](K_{6n+3} - K_{6n-2}) + \log(5)(K_{6n+3} - K_{6n+2}) < 0$$

which is that

$$[2\log(5) - \log(7)]K_{6n+3} + [\log(7) - \log(5)]K_{6n-2} - \log(5)K_{6n+2} < 0$$

After dropping the positive middle term and rearranging we get

$$K_{6n+3} < \frac{\log(5)}{[2\log(5) - \log(7)]}K_{6n+2}$$

so that

$$K_{6n+3} < \frac{\log(7)}{\log(3)}K_{6n+2}$$

however, by condition (1.11) the right hand side is less than K_{6n+3} giving the required contradiction. $\qquad \square$

Consequently, at any length-scale the sets F and G do not both look like the Cantor middle-third set. This lemma gives the following useful corollary:

COROLLARY 2.4. *Every sequence $\{\delta_i\}$ with $\delta_i \to 0$ either contains a subsequence $\{\delta_{i_n}\}$ with each δ_{i_n} corresponding to some stage $j_{\delta_{i_n}} \in (K_{6n+2}, K_{6n+6}]$ in the construction of F or contains a subsequence $\{\delta_{i_m}\}$ corresponding to some stage $j_{\delta_{i_m}} \in (K_{6m+5}, K_{6m+9}]$ in the construction of G.*

Proof. If the sequence $\{\delta_i\}$ did not contain such a subsequence, then there is a $\Delta > 0$ such that each $\delta_i < \Delta$ is neither a length-scale corresponding to some stage $j \in (K_{6n+2}, K_{6n+6}]$ in the construction of F nor a length-scale corresponding to some stage $j \in (K_{6m+5}, K_{6m+9}]$ in the

construction of G. Consequently, δ_i is a length scale corresponding to a stage $j \in (K_{6n}, K_{6n+2}]$ in the construction of F and also a length-scale corresponding to a stage $j \in (K_{6m+3}, K_{6m+5}]$ which, from lemma 2.3, is contradictory. $\qquad\square$

The corresponding lemma that F and G do not both look like the Cantor middle-$\frac{5}{7}$th set at any length-scale and the corresponding corollary for subsequences are proved in a similar way.

LEMMA 2.5.   *No δ is both a length-scale corresponding to some stage in $(K_{6n+1}, K_{6n+3}]$ in the construction of F for some $n \in \mathbb{N}$ and a length-scale corresponding to some stage in $(K_{6m+4}, K_{6m+6}]$ in the construction of G for some $m \in \mathbb{N}$,*

COROLLARY 2.6.   *Every sequence $\{\delta_i\}$ with $\delta_i \to 0$ either contains a subsequence $\{\delta_{i_n}\}$ with each δ_{i_n} corresponding to some stage $j_{\delta_{i_n}} \in (K_{6n+3}, K_{6n+7}]$ in the construction of F or contains a subsequence $\{\delta_{i_m}\}$ corresponding to some stage $j_{\delta_{i_m}} \in (K_{6m}, K_{6m+4}]$ in the construction of G.*

3.   *Calculating box-counting dimensions*

In order to establish the box-counting dimensions of F and G we need the following proposition on the behaviour of the generator-counting functions at the limit:

PROPOSITION 3.1.

$$\frac{f_3\left(K_{6n+l}\right)}{K_{6n+l}} \to \begin{cases} 1 & l = 1 \\ 0 & 0 \le l < 6,\ l \neq 1 \end{cases} \qquad \frac{f_7\left(K_{6n+l}\right)}{K_{6n+l}} \to \begin{cases} 1 & l = 2 \\ 0 & 0 \le l < 6,\ l \neq 2 \end{cases}$$

$$\frac{g_3\left(K_{6m+l}\right)}{K_{6m+l}} \to \begin{cases} 1 & l = 4 \\ 0 & 0 \le l < 6,\ l \neq 4 \end{cases} \qquad \frac{g_7\left(K_{6m+l}\right)}{K_{6m+l}} \to \begin{cases} 1 & l = 5 \\ 0 & 0 \le l < 6,\ l \neq 5 \end{cases}$$

as $n, m \to \infty$.

Proof.   From (2.1) we have

$$\frac{f_3\left(K_{6n+1}\right)}{K_{6n+1}} = \frac{\sum_{i=0}^{n} K_{6i+1} - K_{6i}}{K_{6n+1}} = 1 + \frac{\sum_{i=0}^{n-1} K_{6i+1}}{K_{6n+1}} - \frac{\sum_{i=0}^{n} K_{6i}}{K_{6n+1}}$$

which converges to 1 as $n \to \infty$ by (1.12). By (2.2) we have $f_3\left(K_{6n+l}\right) = f_3\left(K_{6n+1}\right)$ for $2 \le l < 6$ so

$$\frac{f_3\left(K_{6n+l}\right)}{K_{6n+l}} = \frac{f_3\left(K_{6n+1}\right)}{K_{6n+l}} = \frac{\sum_{i=0}^{n} K_{6i+1} - K_{6i}}{K_{6n+l}}$$

which converges to 0 as $n \to \infty$ by (1.12). Similarly, $f_3\left(K_{6n+0}\right) = f_3\left(K_{6(n-1)+1}\right)$ so

$$\frac{f_3\left(K_{6n+0}\right)}{K_{6n+0}} = \frac{f_3\left(K_{6(n-1)+1}\right)}{K_{6n}} = \frac{\sum_{i=0}^{n-1} K_{6i+1} - K_{6i}}{K_{6n}} \to 0$$

The remaining results follow analogously. $\qquad\square$

LEMMA 3.2. $\dim_B\left(F\right) = \dim_B\left(G\right) = \frac{\log(2)}{\log(3)}$.

Proof. First, we show that $\dim_B (F) \leq \frac{\log(2)}{\log(3)}$. Writing j_δ for the stage associated with the length-scale δ we have from (2.5)

$$\frac{\log (N (F, \delta))}{-\log \delta} < \frac{j_\delta \log (2)}{f_3 (j_\delta - 1) [\log (3) - \log (5)] + f_7 (j_\delta - 1) [\log (7) - \log (5)] + (j_\delta - 1) \log (5)}$$
$$\leq \frac{j_\delta \log (2)}{f_3 (j_\delta - 1) [\log (3) - \log (5)] + (j_\delta - 1) \log (5)}$$

and as $f_3 (j_\delta - 1) < j_\delta - 1$

$$< \frac{j_\delta \log (2)}{(j_\delta - 1) [\log (3) - \log (5)] + (j_\delta - 1) \log (5)} = \frac{j_\delta \log (2)}{(j_\delta - 1) \log (3)}$$

which converges to $\frac{\log(2)}{\log(3)}$ as $\delta \to 0$. Similarly we can show $\dim_B (G) \leq \frac{\log(2)}{\log(3)}$

Next, if we take the sequence $\{\delta_n\}$ with

$$-\log \delta_n = f_3 (K_{6n+1}) [\log (3) - \log (5)] + f_7 (K_{6n+1}) [\log (7) - \log (5)] + K_{6n+1} \log (5)$$

we have from (2.5) that $\log (N (F, \delta_n)) = K_{6n+1} \log (2)$. Consequently,

$$\frac{\log (N (F, \delta_n))}{-\log \delta_n} = \frac{K_{6n+1} \log (2)}{f_3 (K_{6n+1}) [\log (3) - \log (5)] + f_7 (K_{6n+1}) [\log (7) - \log (5)] + K_{6n+1} \log (5)}$$
$$= \frac{\log (2)}{\frac{f_3(K_{6n+1})}{K_{6n+1}} [\log (3) - \log (5)] + \frac{f_7(K_{6n+1})}{K_{6n+1}} [\log (7) - \log (5)] + \log (5)}$$
$$\to \frac{\log (2)}{\log (3) - \log (5) + 0 + \log (5)} = \frac{\log (2)}{\log (3)}$$

as $n \to \infty$ by the convergence results (3.1) so that $\dim_B (F) \geq \frac{\log(2)}{\log(3)}$
A similar sequence gives the corresponding inequality for G. $\square$

LEMMA 3.3. $\dim_{LB} (F) = \dim_{LB} (G) = \frac{\log(2)}{\log(7)}$.

Proof. For all $\delta > 0$ the implication (2.5) gives

$$\frac{\log (N (F, \delta))}{-\log \delta} \geq \frac{j_\delta \log (2)}{f_3 (j_\delta) [\log (3) - \log (5)] + f_7 (j_\delta) [\log (7) - \log (5)] + j_\delta \log (5)}$$
$$\geq \frac{j_\delta \log (2)}{f_7 (j_\delta) [\log (7) - \log (5)] + j_\delta \log (5)}$$
$$> \frac{j_\delta \log (2)}{j_\delta [\log (7) - \log (5)] + j_\delta \log (5)} = \frac{\log (2)}{\log (7)}$$

Next, if we take the sequence $\{\delta_n\}$ with

$$-\log \delta_n = f_3 (K_{6n+2}) [\log (3) - \log (5)] + f_7 (K_{6n+2}) [\log (7) - \log (5)] + K_{6n+2} \log (5)$$

we have

$$
\frac{\log\left(N\left(F, \delta_n\right)\right)}{-\log\delta_n} = \frac{K_{6n+2}\log\left(2\right)}{f_3\left(K_{6n+2}\right)\left[\log\left(3\right) - \log\left(5\right)\right] + f_7\left(K_{6n+2}\right)\left[\log\left(7\right) - \log\left(5\right)\right] + K_{6n+2}\log\left(5\right)}
$$

$$
= \frac{\log\left(2\right)}{\frac{f_3\left(K_{6n+2}\right)}{K_{6n+2}}\left[\log\left(3\right) - \log\left(5\right)\right] + \frac{f_7\left(K_{6n+2}\right)}{K_{6n+2}}\left[\log\left(7\right) - \log\left(5\right)\right] + \log\left(5\right)}
$$

$$
\rightarrow \frac{\log\left(2\right)}{0 + \log\left(7\right) - \log\left(5\right) + \log\left(5\right)} = \frac{\log\left(2\right)}{\log\left(7\right)}
$$

as $n \to \infty$ by the convergence results (3.1). Hence $\dim_{LB}\left(F\right) = \frac{\log(2)}{\log(7)}$, and similarly $\dim_{LB}\left(G\right) = \frac{\log(2)}{\log(7)}$. $\qquad\square$

Consequently, both F and G have unequal upper and lower box-counting dimensions:

COROLLARY 3.4. $\dim_{LB}\left(F\right) = \dim_{LB}\left(G\right) < \dim_B\left(F\right) = \dim_B\left(G\right)$.

Whilst the above lemmas demonstrate that for F and G there are sequences of length-scales $\{\delta_n\}$ with $\lim_{n\to\infty}\frac{\log(N(F,\delta_n))}{-\log\delta_n}$ equal to $\frac{\log(2)}{\log(3)}$ or equal to $\frac{\log(2)}{\log(7)}$ we now show that for a large class of sequences (in fact the very sequences that corollaries 2.4 and 2.6 produce) this limit, if it exists, is bounded by $\frac{\log(2)}{\log(5)}$.

LEMMA 3.5.   Suppose $\{\delta_n\}$ is a sequence such that each length-scale δ_n corresponds to the construction of F at some stage $j_n \in \left(K_{6n+2}, K_{6n+6}\right]$, then if the limit exists

$$
\lim_{n\to\infty}\frac{\log\left(N\left(F, \delta_n\right)\right)}{-\log\delta_n} \leq \frac{\log\left(2\right)}{\log\left(5\right)}.
$$

Essentially, these stages are sufficiently far from the range $\left(K_{6n}, K_{6n+1}\right]$ where gen_3 is applied so that the set F does not look like the Cantor middle-third set at these stages. The proof relies on the fact that by stage j_n the generator gen_3 has not been applied for at least the last $K_{6n+2} - K_{6n+1}$ stages.

Proof.   For each $n \in \mathbb{N}$ from (2.5)

$$
\frac{\log\left(N\left(F, \delta_n\right)\right)}{-\log\delta_n} \leq \frac{j_n\log\left(2\right)}{f_3\left(j_n - 1\right)\left[\log\left(3\right) - \log\left(5\right)\right] + f_7\left(j_n - 1\right)\left[\log\left(7\right) - \log\left(5\right)\right] + \left(j_n - 1\right)\log\left(5\right)}
$$

$$
\leq \frac{j_n\log\left(2\right)}{f_3\left(j_n - 1\right)\left[\log\left(3\right) - \log\left(5\right)\right] + \left(j_n - 1\right)\log\left(5\right)}
$$

From (2.2) we have $f_3\left(j_n - 1\right) = f_3\left(K_{6n+1}\right)$ so that

$$
= \frac{j_n\log\left(2\right)}{f_3\left(K_{6n+1}\right)\left[\log\left(3\right) - \log\left(5\right)\right] + \left(j_n - 1\right)\log\left(5\right)}
$$

$$
= \frac{\log\left(2\right)}{\frac{f\left(K_{6n+1}\right)}{j_n}\left[\log\left(3\right) - \log\left(5\right)\right] + \log\left(5\right) - \frac{1}{j_n}\log\left(5\right)}.
$$

Next, as $j_n > K_{6n+2}$

$$\leq \frac{\log(2)}{\frac{f_3(K_{6n+1})}{K_{6n+2}}[\log(3) - \log(5)] + \log(5) - \frac{1}{j_n}\log(5)}$$

$$\rightarrow \frac{\log(2)}{\log(5)}$$

as $n \rightarrow \infty$ by the convergence result (3.1). $\square$

The corresponding result for G, proved in a similar way, is as follows.

LEMMA 3.6.   *Suppose $\{\delta_m\}$ is a sequence such that each length-scale δ_m corresponds to the construction of G at some stage $j_m \in (K_{6m+5}, K_{6m+9}]$, then if the limit exists*

$$\lim_{m \rightarrow \infty} \frac{\log(N(G, \delta_m))}{-\log \delta_m} \leq \frac{\log(2)}{\log(5)}.$$

The following results for lower bounds are also proved similarly.

LEMMA 3.7.   *Suppose $\{\delta_n\}$ is a sequence such that each length-scale δ_n corresponds to the construction of F at some stage $j_n \in (K_{6n+3}, K_{6n+7}]$, then if the limit exists*

$$\lim_{n \rightarrow \infty} \frac{\log(M(F, \delta_n))}{-\log \delta_n} \geq \frac{\log(2)}{\log(5)}$$

and

LEMMA 3.8.   *Suppose $\{\delta_m\}$ is a sequence such that each length-scale δ_m corresponds to the construction of G at some stage $j_m \in (K_{6m}, K_{6m+4}]$, then if the limit exists*

$$\lim_{m \rightarrow \infty} \frac{\log(M(G, \delta_m))}{-\log \delta_m} \geq \frac{\log(2)}{\log(5)}.$$

Finally, we find a bound on the box-counting dimensions of the product $F \times G$.

THEOREM 3.9. $\dim_B (F \times G) \leq \frac{\log(2)}{\log(3)} + \frac{\log(2)}{\log(5)}.$

Proof.   We have from (1.7) that

$$\limsup_{\delta \rightarrow 0} \frac{\log(N(F \times G, \delta))}{-\log \delta} \leq \limsup_{\delta \rightarrow 0} \left[\frac{\log(N(F, \delta))}{-\log \delta} + \frac{\log(N(G, \delta))}{-\log \delta}\right]$$

so it is sufficient to show that the right hand side is no greater than $\frac{\log(2)}{\log(3)} + \frac{\log(2)}{\log(5)}$. Suppose that $\{\delta_i\}$ is a sequence with $\delta_i \rightarrow 0$ such that the limits $\lim_{i \rightarrow \infty} \frac{\log(N(F, \delta_i))}{-\log \delta_i}$ and $\lim_{i \rightarrow \infty} \frac{\log(N(G, \delta_i))}{-\log \delta_i}$ exist. Corollary 2.4 guarantees that this sequence either contains a subsequence $\{\delta_{i_n}\}$ satisfying the hypothesis of lemma 3.5 or contains a subsequence $\{\delta_{i_m}\}$ satisfying the hypothesis of lemma 3.6 so at least one of $\frac{\log(N(F, \delta_{i_n}))}{-\log \delta_{i_n}}$ and $\frac{\log(N(G, \delta_{i_n}))}{-\log \delta_{i_n}}$ converges to $\frac{\log(2)}{\log(5)}$. Using the upper box-counting dimension from lemma 3.2 to bind the other term yields

$$\lim_{n \rightarrow \infty} \frac{\log(N(F, \delta_{i_n}))}{-\log \delta_{i_n}} + \frac{\log(N(G, \delta_{i_n}))}{-\log \delta_{i_n}} \leq \frac{\log(2)}{\log(3)} + \frac{\log(2)}{\log(5)}$$

a bound which also hold for the original sequence $\{\delta_i\}$. As $\{\delta_i\}$ was an arbitrary convergent sequence,

$$\limsup_{\delta \to 0} \left[\frac{\log\left(N\left(F, \delta\right)\right)}{-\log \delta} + \frac{\log\left(N\left(G, \delta\right)\right)}{-\log \delta} \right] \leq \frac{\log\left(2\right)}{\log\left(3\right)} + \frac{\log\left(2\right)}{\log\left(5\right)}$$

$\square$

COROLLARY 3.10. $\dim_B\left(F \times G\right) < \dim_B\left(F\right) + \dim_B\left(G\right)$

THEOREM 3.11. $\dim_{LB}\left(F \times G\right) \geq \frac{\log(2)}{\log(7)} + \frac{\log(2)}{\log(5)}$.

Proof.   From (1.8) we have

$$\liminf_{\delta \to 0} \frac{\log\left(M\left(F \times G, \delta\right)\right)}{-\log \delta} \geq \liminf_{\delta \to 0} \left[\frac{\log\left(M\left(F, \delta\right)\right)}{-\log \delta} + \frac{\log\left(M\left(G, \delta\right)\right)}{-\log \delta} \right]$$

so it is sufficient to prove that the right hand side is no less than $\frac{\log(2)}{\log(7)} + \frac{\log(2)}{\log(5)}$. Suppose that $\{\delta_i\}$ is a sequence with $\delta_i \to 0$ such that the limits $\lim_{i \to \infty} \frac{\log(N(F, \delta_i))}{-\log \delta_i}$ and $\lim_{i \to \infty} \frac{\log(N(G, \delta_i))}{-\log \delta_i}$ exist. In a similar fashion to theorem 3.9, corollary 2.6 and lemmas 3.7 and 3.8 guarantee that at least one of $\lim_{i \to \infty} \frac{\log(M(F, \delta_{i_n}))}{-\log \delta_{i_n}}$ and $\lim_{i \to \infty} \frac{\log(M(G, \delta_{i_n}))}{-\log \delta_{i_n}}$ is no less than $\frac{\log(2)}{\log(5)}$ so

$$\liminf_{\delta \to 0} \left[\frac{\log\left(M\left(F, \delta\right)\right)}{-\log \delta} + \frac{\log\left(M\left(G, \delta\right)\right)}{-\log \delta} \right] \geq \frac{\log\left(2\right)}{\log\left(5\right)} + \frac{\log\left(2\right)}{\log\left(7\right)}$$

$\square$

COROLLARY 3.12. $\dim_{LB}\left(F \times G\right) > \dim_{LB}\left(F\right) + \dim_{LB}\left(G\right)$

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